一、随机过程的概念与基本类型

1、二阶矩过程：$X(t)$ 的均值、方差都存在

性质：由 Cauchy-Schwarz，$R_X(t,s)$、$C_X(t,s)$、$R_{XY}(t,s)$、$C_{XY}(t,s)$ 存在.

1) $R_X(t,s)=E[X(t)\overline{X(s)}]$：共轭对称、非负定

2) $C_X(t,s)=E\{X(t)-E[X(t)]\}\{\overline{X(s)-E[X(s)]}\}$.

3) $R_{XY}(t,s)=E[X(t)\overline{Y(s)}]$.

4) $C_{XY}(t,s)=E\{X(t)-E[X(t)]\}\{\overline{Y(s)-E[Y(s)]}\}$

1) $R_X(t,s)=\overline{E[\overline{X(t)}X(s)]}=\overline{E[X(s)\overline{X(t)}]}=\overline{R_X(s,t)}$.

2) $R_{XY}(t,s)=\overline{E[\overline{X(t)}Y(s)]}=\overline{E[Y(s)\overline{X(t)}]}=\overline{R_{YX}(s,t)}$

3) $C_X(t,s)=E\{X(t)\overline{X(s)}-X(t)\overline{E[X(s)]}-E[X(t)]\overline{X(s)}+E[X(t)]\overline{E[X(s)]}\}$
$=R_X(t,s)-E[X(t)]\overline{E[X(s)]}-E[X(t)]\overline{E[X(s)]}+E[X(t)]\overline{E[X(s)]}$
$=R_X(t,s)-E[X(t)]\overline{E[X(s)]}$.

4) $C_{XY}(t,s)=E\{X(t)\overline{Y(s)}-X(t)\overline{E[Y(s)]}-E[X(t)]\overline{Y(s)}+E[X(t)]\overline{E[Y(s)]}\}$
$=R_{XY}(t,s)-E[X(t)]\overline{E[Y(s)]}$.

2. 平稳过程.

1) 严平稳：$F(x_1,\cdots,x_n;t_1,\cdots,t_n)=F(x_1,\cdots,x_n,t_1+\tau,\cdots,t_n+\tau)$

2) 宽平稳：$E[X(t)]=const$，$R_X(t,s)=R_X(t-s)$.

性质①宽平稳过程是二阶矩过程；
②二阶矩、严平稳 ⇒ 宽平稳.

性质：① $R_X(t-s)=\overline{R_X(s-t)}$，即 $R_X(\tau)=\overline{R_X(-\tau)}$.
② $R_X(0)=E[X^2(t)]\geq E^2[X(t)]$.
③ $|R_X(\tau)|\leq R_X(0)$：由 Cauchy-Schwarz
④ 一元非负定.

3) 联合宽平稳：$R_{XY}(t,s)=R_{XY}(t-s)$.

性质：$R_{XY}(t-s)=\overline{R_{YX}(s-t)}$，即 $R_{XY}(\tau)=\overline{R_{YX}(-\tau)}$.

3. 平稳增量过程：$\forall t<s$，$X(s)-X(t)$ 分布仅取决 $s-t$.

4. 独立增量过程：…$X(t_n)-X(t_{n-1})$ 相互独立.

5. 正交增量过程：$\forall t_1<t_2\leq t_3<t_4$，二阶矩 $X(t)$：
$E[X(t_2)-X(t_1)]\overline{[X(t_4)-X(t_3)]}=0$.

性质：①二阶矩存在，均值为0时，独立增量⇒正交增量.
② $R_X(s,t)=F[\min(s,t)]$，F 单调不减

二、均方微积分：对二阶矩过程.

1. 均方极限：

Cauchy：$E|X_n-X_m|^2\to 0$

Loeve：$E(X_n\overline{X_m})\to const$.

2. 均方连续：$R_X(t,s)$ 在 $t=s$ 上连续

if 宽平稳：只需 $R_X(\tau)$ 在 $\tau=0$ 连续.

3. 均方导数：$\frac{\partial^2}{\partial t\partial s}R_X(t,s)$ 在 (t,t) 处存在且连续

if 宽平稳：只需 $R_X'(\tau)$ 在 $\tau=0$ 可导.

4. 均方积分：$R_X(t,s)$ 对 t,s 可积.

if 宽平稳：只需 $R_X(\tau)$ 可积（连续即可）

性质① $\sqrt{E\left|\int_a^b X(t)\,dt\right|^2}\leq\int_a^b\sqrt{E|X(t)|^2}\,dt$.

② $E[X'(t)\overline{X(s)}]=R_X'(\tau)$
$E[X(t)\overline{X'(s)}]=-R_X'(\tau)$.
$E[X'(t)\overline{X'(s)}]=-R_X''(\tau)$.

三、遍历（各态历经）：对宽平稳且均方可积，以均值遍历为例。

均值遍历：$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\,dt \xrightarrow[T\to\infty]{m.s.} E[X(t)]$

性质：① $E(\langle X(t)\rangle)=\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} E[X(t)]\,dt = \text{const}$

② $D(\langle X(t)\rangle)=\lim_{T\to\infty} E\left|\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} X(t)\,dt-\mu_X\right|^2$

$=\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}\left(1-\frac{|\tau|}{T}\right)C_X(\tau)\,d\tau$

$=0$ —— 充宽平稳。

③ $\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T}C_X(\tau)\,d\tau=0$。

—— 充宽平稳。

④ 充分条件：$C_X(0)<\infty$，$\lim_{\tau\to\infty}C_X(\tau)\to 0$。

—— 宽平稳。

四、K-L展开，主成分分析：对二阶矩过程且均方连续

1) 以 $R_X(t,s)$ 的本征矢 $\{\phi_i(t)\}$ 为基展开 $X(t)$：

$X(t)=\sum_n |\phi_n(t)\rangle\langle\phi_n(t)|X(t)\rangle$

2) 展开系数间的相关性：

$E\{\langle\phi_n(t)|X(t)\rangle\langle X(s)|\phi_m(s)\rangle\}$

$=\langle\phi_n(t)|E[X(t)\rangle\langle X(s)]|\phi_m(s)\rangle$

$=\langle\phi_n(t)|R_X(t,s)|\phi_m(s)\rangle$

$=\lambda_m\langle\phi_n(t)|\phi_m(t)\rangle$

$=\lambda_m\delta_{mn}$。

3) $R(t,s)$ 的特征分解：

$R_X(t,s)=E\{|X(t)\rangle\langle X(s)|\}$

$=E\left\{\sum_n\sum_m|\phi_n(t)\rangle\langle\phi_n(t)|X(t)\rangle\langle X(s)|\phi_m(s)\rangle\langle\phi_m(s)|\right\}$

$=\sum_n\sum_m|\phi_n(t)\rangle\langle\phi_n(t)|R_X(t,s)|\phi_m(s)\rangle\langle\phi_m(s)|$

$=\sum_n\sum_m\lambda_m|\phi_n(t)\rangle\langle\phi_n(t)|\phi_m(t)\rangle\langle\phi_m(s)|$

$=\sum_n\sum_m\lambda_m\delta_{nm}|\phi_n(t)\rangle\langle\phi_m(s)|$

$=\sum_n\lambda_n|\phi_n(t)\rangle\langle\phi_n(s)|$

e.g. 高斯过程的主成分分析：去相关。

（=独立成分分析）。

1) 以 Σ_X 的本征矢 U 为基展开 X。

$X=U^TUX$，系数 $Y=UX$ ~高斯。

2) 展开系数间的相关性。

$\Sigma_Y=E(UX\cdot X^TU^T)$

$=UE(XX^T)U^T$

$=U\Sigma_XU^T$

$=\text{diag}\{\lambda\}U\cdot U^T$

$=\text{diag}\{\lambda\}$

3) Σ_X 的特征分解。

$\Sigma_X=\sum_{k=1}\lambda_kU_kU_k^T$

五、谱分析：以连续 X，连续 t 为例。

宽平稳：$R_X(\tau)\xrightarrow[IFT]{FT}S_X(\omega)$，$R_X(m)\xrightarrow[IDTFT]{DTFT}S_X(e^{j\omega})$。

联合宽平稳：$R_{XY}(\tau)\xrightarrow[IFT]{FT}S_{XY}(\omega)$，$R_{XY}(m)\xrightarrow[IDTFT]{DTFT}S_{XY}(e^{j\omega})$。

性质：① $\overline{S_X(\omega)}=\overline{\int_{-\infty}^{\infty}R_X(\tau)e^{-j\omega\tau}\,d\tau}$

$=\int_{-\infty}^{\infty}\overline{R_X(\tau)}e^{j\omega\tau}\,d\tau$

$=\int_{-\infty}^{\infty}R_X(-\tau)e^{j\omega\tau}\,d\tau$

$=\int_{-\infty}^{\infty}R_X(-\tau)e^{-j\omega(-\tau)}\,d(-\tau)$

$=S_X(\omega)$。

② 由 $R(\tau)$ 的非负定性可证 $S_X(\omega)\geq 0$。

③ 当 $X(t)$ 为实过程时，$R_X(\tau)=R_X(-\tau)$，~~$S_X(\omega)$~~

$S_X(\omega)=2\int_0^{\infty}R_X(\tau)\cos(\omega\tau)\,d\tau$ 为偶函数。

④ $S_{XY}(\omega)=\int_{-\infty}^{\infty}R_{XY}(\tau)e^{-j\omega\tau}\,d\tau$

$$= \int_{-\infty}^{\infty} \overline{R_{YX}(-\tau) e^{-j\omega\tau}} d\tau$$

$$= \int_{-\infty}^{\infty} \overline{R_{YX}(-\tau) e^{j\omega\tau}} d\tau$$

$$= \int_{-\infty}^{\infty} \overline{R_{YX}(-\tau) e^{-j\omega(-\tau)}} d\tau$$

$$= \overline{S_{YX}(\omega)}$$

二阶矩过程通过线性系统.

1) 时变、非宽平稳.

$$Y(t) = \int_{-\infty}^{\infty} h(t,s) X(s) ds.$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(t,s) E[X(s)] ds$$

$$R_{YX}(t,\tau) = E\left\{\int_{-\infty}^{\infty} h(t,s) X(s) ds\, \overline{X(\tau)}\right\}$$

$$= \int_{-\infty}^{\infty} h(t,s) R_X(s,\tau) ds.$$

$$R_Y(u,v) = E\left\{Y(u) \int_{-\infty}^{\infty} \overline{h(v,\tau) X(\tau)} d\tau\right\}$$

$$= \int_{-\infty}^{\infty} \overline{h(v,\tau)} R_{YX}(u,\tau) d\tau$$

$$= \iint_{-\infty}^{\infty} \overline{h(v,\tau)} h(u,s) R_X(s,\tau) ds d\tau.$$

2) 时不变：$h(t,s) = h(t-s)$

3) 宽平稳：$R(t,s) = R(t-s)$，$E[X(t)] = \mu_X$.

4) 时不变、宽平稳：

$$Y(t) = \int_{-\infty}^{\infty} h(t-s) X(s) ds.$$

$$E[Y(t)] = E[X(t)] \int_{-\infty}^{\infty} h(\tau) d\tau = \text{const}$$

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} h(\tau-s) R_X(s) ds = h(\tau) * R_X(\tau)$$

$$R_{XY}(\tau) = \overline{R_{YX}(-\tau)}$$

$$= \int_{-\infty}^{\infty} \overline{h(-\tau-s) \cdot R_X(s)} ds$$

$$= \int_{-\infty}^{\infty} \overline{h(-\tau-s)} R_X(-s) ds$$

$$= \overline{h(-\tau)} * R_X(\tau)$$

$$C_{XY}(\tau) = h(\tau) * C_X(\tau)$$

$$C_Y(\tau) = h(\tau) * \overline{h(-\tau)} * C_X(\tau)$$

$$R_Y(\tau) = h(\tau) * \overline{h(-\tau)} * R_X(\tau) = \overline{h(-\tau)} * R_{YX}(\tau).$$

$$S_{YX}(\omega) = H(\omega) \cdot S_X(\omega); \quad S_{XY}(\omega) = \overline{H(\omega)} S_X(\omega).$$

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega).$$

六. 低通、带通采样：对宽平稳过程：

以连续 x，连续 t 为例.

1. 低通：

$$X(t) \xrightarrow{m.s.} \sum_{k=-\infty}^{\infty} X(kT) \frac{\sin \frac{\pi}{T}(t-kT)}{\frac{\pi}{T}(t-kT)}$$

$$S_X(\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} S_X\left(\frac{\omega - 2k\pi}{T}\right).$$

2. 带通：

$$X(t) \xrightarrow{m.s.} \sum_{k=-\infty}^{\infty} Re\{[X(kT) + j\hat{X}(kT)] e^{-j\frac{\pi}{T}(t-kT)}\} \frac{\sin \frac{\pi}{T}(t-kT)}{\frac{\pi}{T}(t-kT)}$$

其中，$\hat{X}(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{X(\tau)}{t-\tau} d\tau$；$H(\omega) = -j\,\mathrm{sgn}\,\omega$；$h(t) = \frac{1}{\pi t}$

$$R_{\hat{X}}(\tau) = R_X(\tau), \quad S_{\hat{X}}(\omega) = |H(\omega)|^2 S_X(\omega) = S_X(\omega)$$

$$R_{\hat{X}X}(\tau) = E\left\{X(t+\tau) * \frac{1}{\pi(t+\tau)} \cdot \overline{X(t)}\right\} = R_X(\tau) * \frac{1}{\pi\tau}$$

$$R_{X\hat{X}}(\tau) = +\overline{R_{\hat{X}X}(-\tau)} = -R_X(\tau) * \frac{1}{\pi\tau} = -R_{\hat{X}X}(\tau).$$

1) 构造复带通信号 $Y(t) = X(t) + j\hat{X}(t)$，有

$$R_Y(\tau) = E[Y(t+\tau)\overline{Y(t)}]$$

$$= E\{[X(t+\tau) + j\hat{X}(t+\tau)][\overline{X(t) + j\hat{X}(t)}]\}$$

$$= R_X(\tau) - jR_{X\hat{X}}(\tau) + jR_{\hat{X}X}(\tau) + R_{\hat{X}}(\tau)$$

$$= 2R_X(\tau) + 2jR_{\hat{X}X}(\tau).$$

$$S_Y(\omega) = 2S_X(\omega) + 2jS_{\hat{X}X}(\omega)$$

$$= 2S_X(\omega) + 2jS_X(\omega)(-j\,\mathrm{sgn}\,\omega)$$

$$= 2S_X(\omega)(1 + \mathrm{sgn}\,\omega)$$

2) 构造复基带信号 $Z(t) = Y(t) \cdot e^{-j\omega_c t}$，有

$$R_Z(\tau) = E\{Y(t+\tau) e^{-j\omega_c(t+\tau)} \overline{Y(t) e^{-j\omega_c t}}\}$$

$$= R_Y(\tau) e^{-j\omega_c \tau}.$$

$$S_Z(\omega) = S_Y(\omega + \omega_c).$$

3) 分解出复基带信号 $z(t)$ 中的同相分量 $X_I(t)$ 和正交分量 $X_Q(t)$：

$z(t) = Y(t)e^{-j\omega_c t}$

$= (X(t) + j\hat{X}(t))e^{-j\omega_c t}$

$= [X(t)\cos\omega_c t + \hat{X}(t)\sin\omega_c t] + j[-X(t)\sin\omega_c t + \hat{X}(t)\cos\omega_c t]$

$= X_I(t) + jX_Q(t)$

4) $X_I(t)$ 和 $X_Q(t)$ 的统计特性：

$R_{X_I}(\tau) = R_X(\tau)\cos\omega_c\tau + R_{\hat{X}X}(\tau)\sin\omega_c\tau$.

$S_{X_I}(\omega) = \frac{1}{2}[S_X(\omega+\omega_c) + S_X(\omega-\omega_c)] +$ ~~$\frac{1}{2j}[S_X(\omega-\omega_c) - S_X(\omega+\omega_c)]$~~

$\frac{1}{2j}\{S_X(\omega-\omega_c)[-j\,\mathrm{sgn}(\omega-\omega_c)] - S_X(\omega+\omega_c)[-j\,\mathrm{sgn}(\omega+\omega_c)]\}$

$= \frac{1}{2}S_X(\omega+\omega_c)[1+\mathrm{sgn}(\omega+\omega_c)] - \frac{1}{2}S_X(\omega-\omega_c)[1-\mathrm{sgn}(\omega-\omega_c)]$

$R_{X_Q}(\tau) = R_X(\tau)\cos\omega_c\tau + R_{\hat{X}X}(\tau)\sin\omega_c\tau$.

$S_{X_Q}(\omega) = S_{X_I}(\omega)$.

$R_{X_I X_Q}(\tau) = R_X(\tau)\sin\omega_c\tau - R_{\hat{X}X}(\tau)\cos\omega_c\tau$.

$S_{X_I X_Q}(\omega) = \frac{1}{2j}[S_X(\omega-\omega_c) - S_X(\omega+\omega_c)] - \frac{1}{2}\{S_X(\omega-\omega_c)[-j\,\mathrm{sgn}(\omega-\omega_c)] + S_X(\omega+\omega_c)[j\,\mathrm{sgn}(\omega+\omega_c)]\}$

$= \frac{j}{2}S_X(\omega+\omega_c)[1+\mathrm{sgn}(\omega+\omega_c)] - \frac{j}{2}S_X(\omega-\omega_c)[1-\mathrm{sgn}(\omega-\omega_c)]$.

$\because R_{X_I X_Q}(-\tau) = -R_X(-\tau)\sin\omega_c\tau - R_{\hat{X}X}(-\tau)\cos\omega_c\tau$

$= -\overline{R_X(\tau)}\sin\omega_c\tau - \overline{R_{X\hat{X}}(\tau)}\cos\omega_c\tau$

$= -\overline{R_X(\tau)}\sin\omega_c\tau + \overline{R_{\hat{X}X}(\tau)}\cos\omega_c\tau$.

$= -\overline{R_{X_I X_Q}(\tau)}$

$\therefore$ if $X(t)$ 实，则 $R_{X_I X_Q}(-\tau) = -R_{X_I X_Q}(\tau)$，即 $R_{X_I X_Q}(0) = 0$.

$\therefore$ 同一时刻 X_I 和 X_Q 不相关.

又 $\because S_{X_I X_Q}(\omega) = \frac{j}{2}S_X(\omega+\omega_c)\cdot 2U(\omega+\omega_c) - \frac{j}{2}S_X(\omega-\omega_c)\cdot 2U[-(\omega-\omega_c)]$.

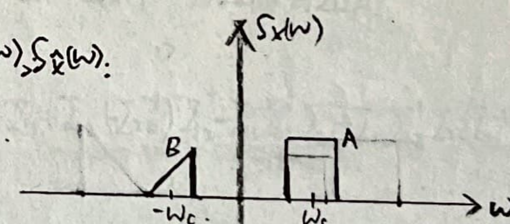
$= j\{S_X(\omega+\omega_c)U(\omega+\omega_c) - S_X(\omega-\omega_c)U[-(\omega-\omega_c)]\}$

$\therefore$ if $S_X(\omega+\omega_c) = S_X(\omega-\omega_c)$，$|\omega| < \omega_c$，则 $S_{X_I X_Q}(\omega) = 0$，即 $R_{X_I X_Q}(\tau) = 0$.

$\therefore$ 实过程 X_I 和 X_Q 不相关.

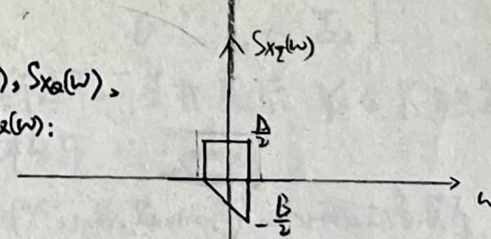
5) 图示：

$S_X(\omega)$, $S_{\hat{X}}(\omega)$:

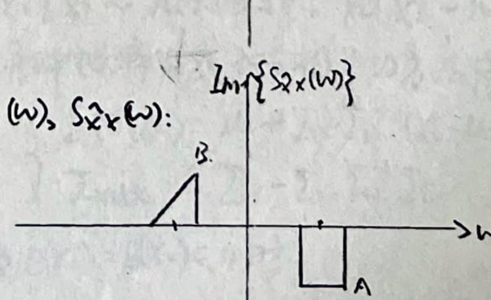


$S_{X_I}(\omega)$, $S_{X_Q}(\omega)$,

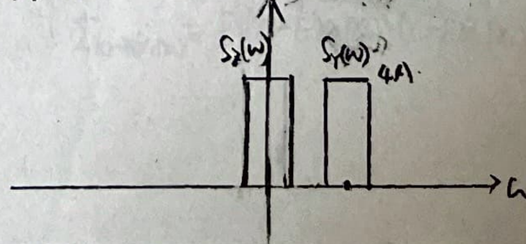
$-j\,S_{X_I X_Q}(\omega)$:



$-S_{X\hat{X}}(\omega)$, $S_{\hat{X}X}(\omega)$:



$S_Y(\omega)$, $S_Z(\omega)$



七. 高斯过程.

1. 密度：$\forall n, \forall t_1,\dots,t_n$，若 $X=(X(t_1)\cdots X(t_n))^T \sim N(\mu,\Sigma)$

则 $f(x)=\frac{1}{(2\pi)^{n/2}\sqrt{|\Sigma|}}\exp\left[-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right]$，$\Sigma$非负定.

2. 特征函数：

$$\Phi_X(\omega)=E\{e^{j\omega^T X}\}=\exp(j\omega^T\mu-\tfrac{1}{2}\omega^T\Sigma\omega).$$

3. 高阶矩：由一、二阶矩完全确定.

$$E(X_1^{k_1}\cdots X_n^{k_n})=\frac{1}{j^{k_1+\cdots+k_n}}\cdot\frac{\partial^{k_1+\cdots+k_n}}{\partial\omega_1^{k_1}\cdots\partial\omega_n^{k_n}}\Phi_X(\omega_1,\dots,\omega_n)\Big|_{\omega^T=0}.$$

e.g. $E(X_1X_2X_3X_4)=E(X_1X_2)E(X_3X_4)+E(X_1X_3)E(X_2X_4)+E(X_1X_4)E(X_2X_3)$.

4. 线性变换：$Y=CX$.

$$\Phi_Y(\omega)=E\{e^{j\omega^T CX}\}=E\{e^{j(C^T\omega)^T X}\}$$
$$=\exp\{j(C^T\omega)^T\mu-\tfrac{1}{2}(C^T\omega)^T\Sigma C^T\omega\}$$
$$=\exp\{j\,\omega^T(C\mu)-\tfrac{1}{2}\omega^T(C\Sigma C^T)\omega\}.$$

1). $X\sim n$元高斯 $\Leftrightarrow \forall C_{n\times1}$，$C^TX\sim$一元高斯

2) $X\sim n$元高斯 $\Rightarrow X$的m元子向量$\sim m$元高斯.

5. 独立性.

1) $X=(X_{1,1\times n_1},X_{2,1\times n_2})^T\sim n_1+n_2$元高斯，$X_1,X_2$独立 $\Leftrightarrow \Sigma_{12}=\Sigma_{21}^T=0$.

2) $X=(X_1\cdots X_n)^T\sim n$元高斯，各元素相互独立 $\Leftrightarrow \Sigma=\mathrm{diag}$.

3) 去相关：$X=(X_{1,1\times n_1},X_{2,1\times n_2})^T$，$X_1,X_2$相关，

$Y=(Y_1,Y_2)^T=\begin{pmatrix}I & A\\ 0 & I\end{pmatrix}\begin{pmatrix}X_1\\X_2\end{pmatrix}$，使$Y_1,Y_2$不相关.

有 $E[(Y_1-EY_1)(Y_2-EY_2)^T]=E\{(X_1+AX_2-EX_1-AEX_2)\cdot(X_2-EX_2)^T\}=E\{(X_1-EX_1)(X_2-EX_2)^T\}+E\{(AX_2-AEX_2)(X_2-EX_2)^T\}=\Sigma_{12}+A\Sigma_{22}\triangleq 0$.

$\therefore A=-\Sigma_{12}\Sigma_{22}^{-1}$.

$\therefore E\{(Y-EY)(Y-EY)^T\}=\begin{bmatrix}\Sigma_{Y11} & \Sigma_{Y12}\\ \Sigma_{Y21} & \Sigma_{Y22}\end{bmatrix}=$

$$\begin{bmatrix}E(X_1+AX_2-EX_1-AEX_2)(X_1+AX_2-EX_1-AEX_2)^T & 0\\ 0 & E(X_2-EX_2)(X_2-EX_2)^T\end{bmatrix}$$

$$=\begin{bmatrix}E[(X_1-EX_1)+A(X_2-EX_2)][(X_1-EX_1)+A(X_2-EX_2)]^T & 0\\ 0 & \Sigma_{22}\end{bmatrix}$$

$$=\begin{bmatrix}\Sigma_{11}+\Sigma_{12}A^T+A\Sigma_{21}+A\Sigma_{22}A^T & 0\\ 0 & \Sigma_{22}\end{bmatrix}$$

$$=\begin{bmatrix}\Sigma_{11}+\Sigma_{12}(-\Sigma_{22}^{-1})^T\Sigma_{12}^T-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}+\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{22}(\Sigma_{22}^{-1})^T\Sigma_{12}^T & 0\\ 0 & \Sigma_{22}\end{bmatrix}$$

$$=\begin{bmatrix}\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21} & 0\\ 0 & \Sigma_{22}\end{bmatrix}.$$

对高斯过程，去相关后 Y_1 与 Y_2 独立.

6. 条件分布.

$X=(X_{1,1\times n_1},X_{2,1\times n_2})^T\sim n_1+n_2$元高斯 $\Rightarrow$

$X_1|X_2\sim n_1$元高斯，$X_2|X_1\sim n_2$元高斯.

用条件概率公式和去相关的方法得

$$\begin{cases}E(X_1|X_2)=\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(X_2-\mu_2)\\ \Sigma_{X_1|X_2}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\end{cases}$$

当 $E(X_1)=E(X_2)=0$ 时.

$$\begin{cases}\Sigma_{X_1|X_2}=E\{(X_1-E(X_1|X_2))(X_1-E(X_1|X_2))^T|X_2\}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\\ \Sigma_{X_1-E(X_1|X_2)}=E\{(X_1-E(X_1|X_2))(X_1-E(X_1|X_2))^T\}=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}\end{cases}$$

7. Gauss-Markov性.

1). $X(t)$ Gauss:

$X(t)$ Markov $\Leftrightarrow E[X(t_n)|X(t_{n-1})\cdots X(t_1)] = E[X(t_n)|X(t_{n-1})]$

2) $X(t)$ Gauss, $\mu_x=0$:

$X(t)$ Markov $\Leftrightarrow k\le n-1$, $E\{[X_n - E(X_n|X_{n-1})]X_k\}=0$.

3) $X(t)$ Gauss, $\mu_x=0$, $X(t)$实:

$X(t)$ Markov $\Leftrightarrow C_x(t_1,t_3)=\dfrac{C_x(t_1,t_2)C_x(t_2,t_3)}{C_x(t_2,t_2)}$

$t_1\le t_2\le t_3$. 用去相关的方法证明.

8. 高斯过程通过非线性系统:

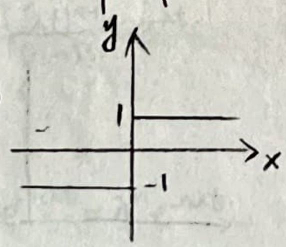
对输入为零均值宽平稳的高斯过程.

1) 限幅器

$f(y) = \frac{1}{2}\delta(y+1)+\frac{1}{2}\delta(y-1)$

$E[Y(t)] = 0$

$R_Y(t,s) = \frac{2}{\pi}\arcsin\left(\frac{R_x(t-s)}{\sigma^2}\right)$

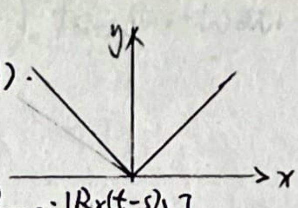


2) 全波检波.

$f(y) = \frac{2}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)u(y)$.

$E[Y(t)] = \sqrt{\frac{2}{\pi}}\sigma$

$R_Y(t,s)=\frac{2\sigma^2}{\pi}\left[\sqrt{1-\frac{R_x^2(t-s)}{\sigma^2}}+\frac{R_x(t-s)}{\sigma^2}\arcsin\left(\frac{R_x(t-s)}{\sigma^2}\right)\right]$.

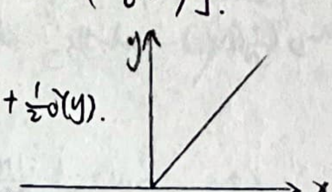


3) 半线性检波.

$f(y)=\frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{y^2}{2\sigma^2}\right)u(y)+\frac{1}{2}\delta(y)$.

$E[Y(t)]=\frac{1}{\sqrt{2\pi}}\sigma$.

$R_Y(t,s)=\frac{\sigma^2}{2\pi}\left\{\sqrt{1-\frac{R_x^2(t-s)}{\sigma^2}}+\frac{R_x(t-s)}{\sigma^2}\left[\frac{\pi}{2}+\arcsin\left(\frac{R_x(t-s)}{\sigma^2}\right)\right]\right\}$.

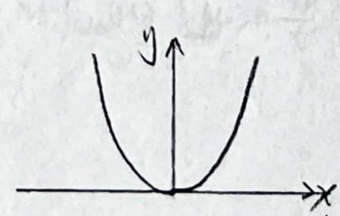


4) 平方律检波.

$f(y)=\frac{1}{\sqrt{2\pi}\sigma}e^{\left(-\frac{y}{2\sigma^2}\right)}\frac{1}{\sqrt{y}}$

$E[Y(t)]=\sigma^2$

$R_Y(t,s)=\sigma^4+2R_x^2(t-s)$:



9. 窄带高斯过程: 对实 $X(t)$: (X_I、X_Q不相关=独立)

1) 零均值.

$Z(t)=X_I(t)\cos\omega_c t - X_Q(t)\sin\omega_c t$

$=\sqrt{X_I^2(t)+X_Q^2(t)}\cos(\omega_c t+\theta(t))$

$=V\cos(\omega_c t+\theta)$.

$f_{V\theta}(v,\theta)=\frac{v}{2\pi\sigma^2}\exp\left(-\frac{v^2}{2\sigma^2}\right)=f(v)\cdot f(\theta)$.

$f(v)=\frac{v}{\sigma^2}\exp\left(-\frac{v^2}{2\sigma^2}\right)$: 瑞利

$f(\theta)=\frac{1}{2\pi}$: 均匀.

but $f(v_1,v_2,\theta_1,\theta_2)\ne f(v_1,v_2)f(\theta_1,\theta_2)$: 渐近独立

2) 加含均匀分布随机相位正弦信号.

$Z(t)=X_I(t)\cos\omega_c t - X_Q(t)\sin\omega_c t + P\sin(\omega_c t+\varphi)$.

$=[P\sin\varphi+X_I(t)]\cos\omega_c t-[X_Q(t)-P\cos\varphi]\sin\omega_c t$

$=V\cos(\omega_c t+\theta)$.

$f(v,\theta,\varphi)=\frac{v}{4\pi^2\sigma^2}\exp\left\{-\frac{v^2+P^2+2vP\sin(\theta-\varphi)}{2\sigma^2}\right\}=f(v)f(\theta)f(\varphi)$.

$f(v)=\frac{v}{\sigma^2}\exp\left(-\frac{v^2+P^2}{2\sigma^2}\right)\cdot I_0\left(\frac{Pv}{\sigma^2}\right)$: 莱斯.

$f(\theta)=f(\varphi)=\frac{1}{2\pi}$: 均匀.

$\begin{cases}P\ll\sigma: & f(v)\approx v\exp\left(-\frac{v^2}{2}\right): 瑞利\\ P\gg\sigma: & f(v)\approx\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{(v-P/\sigma)^2}{2}\right\}: 高斯\end{cases}$.

10. 泊松过程.

1. 泊松(标准): 单一事件、独立增量、平稳增量.

$P(N(t)=n)=\frac{(\lambda t)^n}{n!}e^{-\lambda t}$.

$E[N(t)]=\sum_{n=0}^{\infty}n\cdot\frac{(\lambda t)^n}{n!}e^{-\lambda t}=\lambda t$.

$R_N(t,s)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{\lambda^n s^m(t-s)^{n-m}}{m!(n-m)!}m\cdot n\, e^{-\lambda t}$

$=\lambda^2 st+\lambda\min(s,t)$.

$C_N(t,s)=R_N(t,s)-E[N(t)]E[N(s)]=\lambda\min(s,t)$

$D[N(t)]=C_N(t,t)=\lambda t$.

$G_N(z)=e^{\lambda t(z-1)}=\sum_{n=0}^{\infty}\frac{(\lambda t)^n}{n!}e^{-\lambda t}\cdot z^n$

1) 事件间隔 T_n:

$\because P\{T_n > t\} = P\{N(t)=0\} = e^{-\lambda t}$

$\therefore F_{T_n}(t) = P\{T_n \le t\} = 1 - e^{-\lambda t}$

$\therefore f_{T_n}(t) = \lambda e^{-\lambda t}$，$t \ge 0$

由 $N(t)$ 的独立增量性可证 $T_1 \cdots T_n$ 独立同分布

$E(T_n) = \frac{1}{\lambda}$，$D(T_n) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$.

$\Phi_{T_n}(\omega) = \frac{\lambda}{\lambda - j\omega}$.

2) 等待时间 S_n:

a. $\because \Phi_{S_n}(\omega) = \prod_{i=1}^{n} \Phi_{T_i}(\omega) = \frac{\lambda^n}{(\lambda - j\omega)^n}$

$\therefore f_{S_n}(t) = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t}$

b. $\because P\{S_n \le t\} = P\{N(t) \ge n\} = \sum_{k=n}^{+\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

$\therefore f_{S_n}(t) = \frac{d}{dt} P\{S_n \le t\} = \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t}$.

c. $\because \begin{pmatrix} S_1 \\ \vdots \\ S_n \end{pmatrix} = \begin{pmatrix} 1 & & \\ \vdots & \ddots & \\ 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} T_1 \\ \vdots \\ T_n \end{pmatrix}$

$\therefore f_{S_1 \cdots S_n}(t_1, \cdots, t_n) = |J| \cdot \lambda^n e^{-n\lambda t} = \lambda^n e^{-n\lambda t}$.

$\therefore f_{S_n}(t) = \int_0^t dt_{n-1} \int_0^{t_{n-1}} dt_{n-2} \cdots \int_0^{t_2} f_{S_1 \cdots S_n}(t_1, \cdots, t_n) dt_1$

$= \frac{(\lambda t)^{n-1}}{(n-1)!} \lambda e^{-\lambda t}$.

3) $S_1 \sim S_n$ 的条件分布.

a. $\because P\{u_1 \le S_1 \le u_1 + h_1, \cdots, u_n \le S_n \le u_n + h_n \mid N(t) = n\}$

$= \lambda h_1 e^{-\lambda h_1} \cdots \lambda h_n e^{-\lambda h_n} \cdot e^{-\lambda(t - h_1 - \cdots - h_n)} / \frac{(\lambda t)^n}{n!} e^{-\lambda t}$

$= \frac{n!}{t^n} h_1 \cdots h_n$.

$\therefore f_{S_1 \cdots S_n | N(t)=n}(u_1 \cdots u_n) = \frac{n!}{t^n}$ $(0 \le u_1 < u_2 \cdots \le t)$.

b. $\because S_1 \cdots S_n$ 是 t 内 n 个事件的顺序统计量

$\therefore f_{S_1 \cdots S_n | N(t)=n}(u_1 \cdots u_n) = n! f_{S_1|N}(u_1) \cdots f_{S_n|N}(u_n) = \frac{n!}{t^n}$.

4). S_k 的条件分布

$\because P\{u_k \le S_k \le u_k + h_k \mid N(t) = n\}$

$= \frac{(\lambda u_k)^{k-1}}{(k-1)!} e^{-\lambda u_k} \cdot \lambda h_k e^{-\lambda h_k} \cdot \frac{[\lambda(t - u_k - h_k)]^{n-k}}{(n-k)!} e^{-\lambda(t - u_k - h_k)}$

$\div \frac{(\lambda t)^n}{n!} e^{-\lambda t}$

$= k C_n^k \frac{u_k^{k-1}}{t^k} (1 - \frac{u_k - h_k}{t})^{n-k} h_k$.

$\therefore f_{S_k | N(t)=n}(s) = k C_n^k \frac{s^{k-1}}{t^k} (1 - \frac{s}{t})^{n-k}$.

5) 两次 A 事件 (μ) 间隔内，B 事件 (λ) 发生次数：

$\because P\{N_B(t) = k \mid T_A = t\} = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$

$\therefore P\{N_B(t) = k\} = \int_0^{+\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \cdot \mu e^{-\mu t} dt$

$= \int_0^{+\infty} \frac{\lambda^k \mu}{k!} t^k e^{-(\lambda + \mu)t} dt$

$= (\frac{\mu}{\mu + \lambda})(\frac{\lambda}{\mu + \lambda})^k$

6). 独立的 A (μ) B (λ) 事件共发生 n 次下，A 发生次数：

$P\{N_A(t) = k \mid N_A(t) + N_B(t) = n\} = \frac{P\{N_A(t) = k, N_A + N_B = n\}}{P\{N_A + N_B = n\}}$

$= \frac{P\{N_A(t) = k\} P\{N_B(t) = n - k\}}{P\{N_A + N_B = n\}}$

$= C_n^k (\frac{\mu}{\lambda + \mu})^k (\frac{\lambda}{\lambda + \mu})^{n-k}$：二项分布

$\therefore E\{N_A(t) \mid N_A + N_B = n\} = \sum_{k=0}^{\infty} k C_n^k (\frac{\mu}{\lambda + \mu})^k (\frac{\lambda}{\lambda + \mu})^{n-k}$

$= n \frac{\mu}{\lambda + \mu}$.

2. 非齐次泊松：非平稳增量.

$P\{N(t-s) = n\} = \frac{(\int_s^t \lambda(u) du)^n}{n!} e^{-\int_s^t \lambda(u) du}$.

$E\{N(t-s)\} = \int_s^t \lambda(u) du$.

$D\{N(t-s)\} = \int_s^t \lambda(u) du$、

证法与标准泊松相同.

3. 复合泊松：非单一事件.

$Y(t)=\sum_{k=1}^{N(t)} h_k$，$h_k$ 独立同分布，与 $N(t)$ 独立.

$G_Y(z)=E\{z^Y\}=E_N\{E\{z^{h_1+\cdots+h_N}|N=n\}\}$

$=E_N\{G_h^N(z)\}$

$=e^{\lambda t[G_h(z)-1]}$

a. $E(Y)=G_Y'(z)|_{z=1}=E(N)E(h_i)=\lambda t E(h_i)$.

$E(Y^2)=G_Y''(z)+G_Y'(z)|_{z=1}=\lambda t[E(h_i^2)+E(h_i)]$

$D(Y)=E(Y^2)-E(Y)=\lambda t E\{h_i^2\}$.

b. $E(Y)=E_N E\{\sum_{i=1}^N h_i|N(t)=n\}=E_N\{NE(h_i)\}=\lambda t E\{h_i\}$

$D(Y)=E_N E[\sum_{i=1}^N h_i-\lambda t E(h_i)]^2$

$=E_N E[\sum_{i=1}^N[h_i-E\{h_i\}]+(N-\lambda t)E(h_i)]^2$

$=E_N\{E\sum_{i=1}^N[h_i-E(h_i)]^2+$交叉项$+E[(N-\lambda t)E(h_i)]^2\}$

$=E_N[ND(h_i)+(N-\lambda t)^2E^2(h_i)]$

$=\lambda t D(h_i)+\lambda t E^2(h_i)$

$=\lambda t E(h_i^2)$.

1) 独立泊松之和：$\Phi_{N_1+N_2}(\omega)=e^{(\lambda_1+\lambda_2)t(e^{j\omega}-1)}$.

2) 独立泊松之差：$\Phi_{N_1-N_2}(\omega)=e^{(\lambda_1+\lambda_2)t(\Phi_h(\omega)-1)}$.

$\Phi_h(\omega)=\frac{\lambda_1}{\lambda_1+\lambda_2}e^{j\omega}+\frac{\lambda_2}{\lambda_1+\lambda_2}e^{-j\omega}$：两点分布.

3) 泊松事件分类：按概率的 0-1 变量分类，每类仍泊松.

4. 随机参数：非独立增量.

$P\{Y(t)=k\}=\int_0^\infty \frac{(\lambda t)^n}{n!}e^{-\lambda t}f(\lambda)d\lambda$，$\lambda$ 与 $N(t)$ 独立.

$G_Y(z)=E_\lambda E\{z^Y|\Lambda=\lambda\}=\int_0^\infty e^{\lambda t(z-1)}f(\lambda)d\lambda$

$E(Y)=\frac{\partial}{\partial z}G_Y(z)|_{z=1}=\int_0^\infty \lambda t f(\lambda)d\lambda$.

1) 参数 λ 的后验分布

∵ $P\{\Lambda\le x|Y(t)=n\}=P\{\Lambda\le x,Y(t)=n\}/P\{Y(t)=n\}$

$=\int_0^x \frac{(\lambda t)^n}{n!}e^{-\lambda t}f(\lambda)d\lambda\Big/\int_0^\infty \frac{(\lambda t)^n}{n!}e^{-\lambda t}f(\lambda)d\lambda$

∴ $f\{x|Y(t)=n\}=(xt)^n e^{-xt}f(x)\Big/\int_0^\infty(\lambda t)^n e^{-\lambda t}f(\lambda)d\lambda$.

2) 事件间隔的后验分布.

$P\{T_{n+1}\le x|Y(t)=n\}=P\{T_{n+1}\le x,Y(t)=n\}/P\{Y(t)=n\}$

$=\int_0^\infty P\{T_{n+1}\le x,Y(t)=n|\Lambda=\lambda\}f(\lambda)d\lambda\Big/P\{Y(t)=n\}$

$=\int_0^\infty P\{Y(x)\ge1|\Lambda=\lambda\}P\{Y(t)=n|\Lambda=\lambda\}f(\lambda)d\lambda\Big/P\{Y(t)=n\}$

$=\int_0^\infty(1-e^{-\lambda x})(\lambda t)^n e^{-\lambda t}f(\lambda)d\lambda\Big/\int_0^\infty(\lambda t)^n e^{-\lambda t}f(\lambda)d\lambda$.

3) 等待时间的后验分布

$P\{S_{n+1}\le x|Y(t)=n\}=P\{S_{n+1}\le x,Y(t)=n\}/P\{Y(t)=n\}$

$=\int_0^\infty P\{S_{n+1}\le x,Y(t)=n|\Lambda=\lambda\}f(\lambda)d\lambda\Big/P\{Y(t)=n\}$

$=\int_0^\infty P\{Y(x-t)\ge1|\Lambda=\lambda\}P\{Y(t)=n|\Lambda=\lambda\}f(\lambda)d\lambda\Big/P\{Y(t)=n\}$

$=\int_0^\infty(1-e^{-\lambda(x-t)})(\lambda t)^n e^{-\lambda t}f(\lambda)d\lambda\Big/\int_0^\infty(\lambda t)^n e^{-\lambda t}f(\lambda)d\lambda$

5. 过滤泊松：非平稳、非独立.（若幅度非0则非单一事件）

$Y(t)=\sum_{k=1}^{N(t)}h(t-S_k)$

$\Phi_Y(\omega)=E_N E\{e^{j\omega\sum_{k=1}^{N(t)}h(t-S_k)}|N(t)=n\}$

$=E_N\{\int_0^t\cdots\int_0^{s_3}\int_0^{s_2}e^{j\omega\sum_{k=1}^N h(t-S_k)}\frac{n!}{t^n}ds_1\cdots ds_N\}$

$=E_N\{\int_0^t\cdots\int_0^t\int_0^t e^{j\omega\sum_{k=1}^N h(t-V_k)}\frac{1}{t^n}|J|dv_1\cdots dv_N\}$

$=E_N\{\int_0^t\frac{1}{t}e^{j\omega h(t-\tau)}d\tau\}^N$

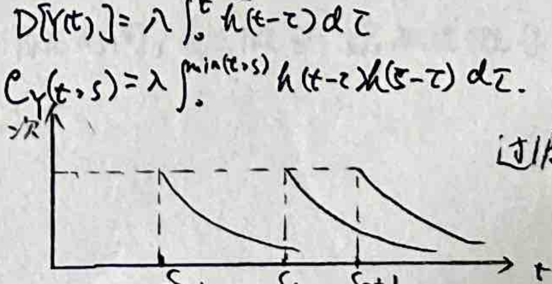
$=\exp\{\lambda t[\int_0^t\frac{1}{t}e^{j\omega h(t-\tau)}d\tau-1]\}$

∴ $E[Y(t)]=\frac{1}{j}\frac{\partial}{\partial\omega}\Phi_Y(\omega)|_{\omega=0}=\lambda\int_0^t h(t-\tau)d\tau$

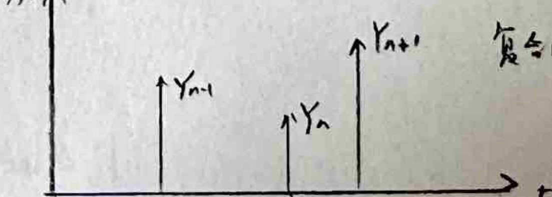
$E[Y^2(t)]=\frac{1}{j^2}\frac{\partial^2}{\partial\omega^2}\Phi_Y(\omega)|_{\omega=0}=\lambda\int_0^t h^2(t-\tau)d\tau+[\lambda\int_0^t h(t-\tau)d\tau]^2$

∴ $D[Y(t)]=\lambda\int_0^t h^2(t-\tau)d\tau$

$C_Y(t,s)=\lambda\int_0^{\min(t,s)}h(t-\tau)h(s-\tau)d\tau$.



过滤泊松



复合泊松.

九. Markov过程：以离散t为例

1. 定义：$\forall n, P\{X_n=i_n \mid X_{n-1}=i_{n-1},\cdots,X_0=i_0\}$

$= P\{X_n=i_n \mid X_{n-1}=i_{n-1}\}$.

1). A：过去；B：现在；C：将来

$P\{C|B,A\} = P\{C|B\}$

$P\{CA|B\} = P\{C|AB\}\cdot P\{A|B\} = P\{C|B\}\cdot P\{A|B\}$

2) $P\{X_{n+1}=j \mid (X_{n-1}\cdots X_0)\in A, X_n=i\} = P\{X_{n+1}=j \mid X_n=i\}$

3) $P\{X_{n+1}\in B \mid (X_{n-1}\cdots X_0)\in A, X_n=i\} = P\{X_{n+1}\in B \mid X_n=i\}$.

4) $P\{X_{n+1}=j \mid (X_{n-1}\cdots X_0)\in A, X_n\in B\} \neq P\{X_{n+1}=j \mid X_n\in B\}$.

2. 齐次Markov

1) $P_{ij}^{(n)} = P\{X_{m+n}=j \mid X_m=i\} \Rightarrow$

$\forall m\ge 0, P_{ij}^{(n)} = P\{X_{m+n}=j \mid X_m=i\}$.

2) $P_{ij}(m,n) = \sum_k P_{ik}(m,l)P_{kj}(l,n) \Rightarrow$

$P_{ij}^{(m+n)} = \sum_k P_{ik}^{(m)} P_{kj}^{(n)}$.

3) $\{Z_n\}$独立，$X_{n+1}=f(X_n, Z_{n+1})$：Markov $\Rightarrow$

$\{Z_n\}$独立同分布，$X_{n+1}=f(X_n,Z_{n+1})$：齐次Markov.

3. 链的转移流程.

1) 可达（对状态）：$i\to j$　传递性.

2) 互通（对状态）：$i\leftrightarrow j$

3) 不可约（对集）：$\forall i,j\in C, i\leftrightarrow j$.

4) 闭集（对集）：$\forall i\in C, j\notin C, i\not\to j$，可能可约

4. 链的转移结果（设$i\leftrightarrow j$）.

1) n步首达概率：$f_{ij}^{(n)}$

2) 迟早到达概率：$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^{(n)}$.

a. $P_{ij}^{(n)} = \sum_{l=1}^{n} P_{jj}^{(n-l)} f_{ij}^{(l)}$，$f_{ij} \ge P_{ij}^{(n)} \ge f_{ij}^{(n)}$.

$$\begin{cases} P_{ij}(z) = P_{jj}(z)\cdot F_{ij}(z), & i\neq j \\ P_{jj}(z) = 1 + F_{jj}(z)P_{jj}(z) \Rightarrow \sum_{n=0}^{\infty} P_{jj}^{(n)} = \frac{1}{1-f_{jj}} = \sum_{m=1}^{\infty} m f_{jj}^{m-1}(1-f_{jj}) = E\{\sum_{n=0}^{\infty} A_n \mid X_0=j\}. \end{cases}$$

b. $q_{ij}(m) \triangleq P\{\sum_{k=0}^{\infty} A_k \ge m \mid X_0=i\}$

$$\begin{cases} q_{jj}(m) = (f_{jj})^{m-1} = \sum_{r\neq j} P_{jr}\, q_{rj}(m-1) + P_{jj} \\ q_{ij}(m) = f_{ij}\, q_{jj}(m) & i\neq j. \end{cases}$$

3) 非常返：

$f_{jj}\cdot f_{ij} < 1$，$\sum_{n=0}^{\infty} P_{jj}^{(n)} < \infty$，$\lim_{m\to\infty} q_{ij}(m) = \lim_{m\to\infty} q_{jj}(m) = 0$

4) 常返：

$f_{jj} = f_{ij} = 1$，$\sum_{n=0}^{\infty} P_{ij}^{(n)} = \infty$，$\lim_{m\to\infty} q_{ij}(m) = \lim_{m\to\infty} q_{jj}(m) = 1$.

a. $f_{jj}^{(n)}$有概率分布的特性.

b. $\mu_j \triangleq E\{T_{jj}\} = \sum_{n=1}^{\infty} n f_{jj}^{(n)}$：平均首达步数.

$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} P_{ij}^{(k)} = \lim_{n\to\infty} P_{jj}^{(n)} = \frac{1}{\mu_j}$：弱遍历性.

5) 正常返（对互通的状态）：

$\sum_{k=0}^{n-1} P_{ij}^{(k)} = O(n)$，$\lim_{n\to\infty} P_{jj}^{(n)} \neq 0$，$\mu_j < \infty$.

6) 零常返（对互通的状态）：

$\sum_{k=0}^{n-1} P_{jj}^{(k)} = o(n)$，$\lim_{n\to\infty} P_{jj}^{(n)} = 0$，$\mu_j = \infty$.

反证：没有有限状态的零常返. 根据$\sum_{j=1}^{n} P_{ij}^{(n)} = 1$.

对正常返：

$$\begin{cases} P_{ij}^{(n+1)} = \sum_k P_{ik}^{(n)} P_{kj} \xrightarrow{n\to\infty} \pi_j = \sum_k \pi_k P_{kj} \\ 1 = \sum_j P_{kj}^{(n)} \xrightarrow{n\to\infty} 1 = \sum_j \pi_j \end{cases}$$

对非常返：

首达时间 / 首达概率. 按物理概念算.

$\alpha e^{-\alpha\|\tau\|}$	$\dfrac{2}{(\omega/\alpha)^2+1}$
$\delta(\tau)$	1
$\cos\omega_0\tau$	$\frac{1}{2}[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
$\dfrac{\sin\omega_0\tau}{\pi\tau}$	$u(\omega+\omega_0)-u(\omega-\omega_0)$
$1-\dfrac{\|\tau\|}{T_0},\ \|\tau\|\le T_0$	$T_0\,Sa^2(\frac{\omega T_0}{2})$
$\alpha e^{-\alpha\|\tau\|}\cos\omega_0\tau$	$\dfrac{1}{(\frac{\omega-\omega_0}{\alpha})^2+1}+\dfrac{1}{(\frac{\omega+\omega_0}{\alpha})^2+1}$
$e^{-\alpha\tau^2}$	$\sqrt{\frac{\pi}{\alpha}}\,e^{-\frac{\omega^2}{4\alpha}}$
$e^{-\alpha\tau^2}\cos\omega_0\tau$	$\frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\left[e^{-\frac{(\omega-\omega_0)^2}{4\alpha}}+e^{-\frac{(\omega+\omega_0)^2}{4\alpha}}\right]$

Poisson:

矩	阶幂			
1	1			
2	1	1		
3	1	3	1	
4	1	6	7	1

$e^{j\omega T}+e^{-j\omega T}-2$